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MOTION OF A RIGID BODY UNDER THE EFFECT OF ROTATION OF AN INTERNAL FLYWHEEL

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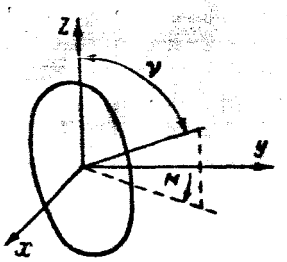
The solution of the problem of the finite rotation of an asymmetric solid under the effect of rotation of an internal flywheel whose axis is arbitrarily fixed in the body is examined. It is shown that, in the general case of rotation of the flywheel about an arbitrary axis, the supporting body executes pure rotation about another axis known as the associate axis. Formulas are derived for determining the direction of the axis of the flywheel fixed in the body with respect to a given magnitude of the three-dimensional rotation of the supporting body.

Let us assume that the direction of the axis of rotation of the flywheel, relative to the principal axes of inertia x, y, z of the body is determined (see diagram) by the angles μ and ν, that the angular velocity of the flywheel relative to the housing is $\Omega(t)$, and that its moment of inertia relative to the angle of rotation is J. Furthermore, we will consider that originally the supporting body and the flywheel are fixed, i.e., that their total kinetic momentum is equal to zero. Then from the law of the conservation of kinetic momentum for the supporting body-flywheel system we have

$$\begin{aligned} A\omega_x + J\Omega(t)\sin\nu\sin\mu &= 0 \\ B\omega_y + J\Omega(t)\sin\nu\cos\mu &= 0 \\ C\omega_z + J\Omega(t)\cos\nu &= 0 \end{aligned} \quad , \tag{1}$$

where A, B, C are the principal moments of inertia of the supporting body together with the flywheel, and ω_x, ω_y, ω_z are the projections of the vector of

angular velocity of the body onto its principal axes x, y, z. To find the motion of the body under the effect of rotation of the flywheel, we can replace the projections of the angular velocity in eqs.(1) by their expressions in terms of angles and derivatives of angles determining the orientation of the support-



ing body in space. In the problem under consideration, it is more convenient to determine the orientation of the body by means of the Cayley-Klein parameters α and β which are connected with the Euler angles θ, ψ, φ by the relations

$$\alpha = \cos {}^1/_2\theta e^{{}^1/_2 i(\psi+\varphi)}, \qquad \beta = i \sin {}^1/_2\theta e^{{}^1/_2 i(\psi-\varphi)}. \tag{2}$$

The equations describing the motion of the body can be written in the form (Bibl.1)

$$\alpha' = {}^1/_2 i\omega_z \alpha + {}^1/_2 (\omega_y + i\omega_x)\beta, \qquad \beta' = -{}^1/_2 i\omega_z \beta - {}^1/_2 (\omega_y - i\omega_x)\alpha. \tag{3}$$

Substituting here the values ω_x, ω_y, ω_z expressed in eqs.(1), we obtain

$$\begin{aligned} \alpha' &= -i\frac{J\Omega\cos\nu}{2C}\alpha - \frac{J\Omega\sin\nu}{2}\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)\beta \\ \beta' &= i\frac{J\Omega\cos\nu}{2C}\beta + \frac{J\Omega\sin\nu}{2}\left(\frac{\cos\mu}{B} - i\frac{\sin\mu}{A}\right)\alpha. \end{aligned} \tag{4}$$

It follows from the form of this system that the rotation of the housing of the supporting body is completely determined by the angle of rotation of the flywheel and does not depend upon the character of the change in the velocity of its rotation. Formally, this implies that in the described system we can neglect the time t and introduce, as an independent variable, the angle of rotation

of the flywheel

$$\tau = \int_0^t \Omega(t)\, dt \ . \tag{5}$$

After this, the system (4) takes the following form:

$$\begin{aligned} \alpha' &= -i\frac{J\cos\nu}{2C}\alpha - \frac{J\sin\nu}{2}\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)\beta \\ \beta' &= i\frac{J\cos\nu}{2C}\beta + \frac{J\sin\nu}{2}\left(\frac{\cos\mu}{B} - i\frac{\sin\mu}{A}\right)\alpha \ . \end{aligned} \tag{6}$$

Here the prime denotes differentiation with respect to τ. Equation (6) represents a homogeneous linear system of equations with constant coefficients.

Its solution, as usual, will be sought in the form /172

$$\alpha = Me^{\gamma\tau}, \qquad \beta = Ne^{\gamma\tau} \ . \tag{7}$$

Substituting the characteristic equation of system (6), we find

$$\gamma = \pm\frac{i}{2}\left[\left(\frac{J\cos\nu}{C}\right)^2 + \left(\frac{J\sin\nu\cos\mu}{B}\right)^2 + \left(\frac{J\sin\nu\sin\mu}{A}\right)^2\right]^{1/2} = \pm\frac{i\lambda}{2} \ . \tag{8}$$

We easily note [returning to the initial system (1)] that the expression under the radical is none other than the ratio ω^2/Ω^2, where $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, so that from eq.(8) we have a relationship valid for any instant of time

$$\omega / \Omega = \lambda = \text{const} \ . \tag{9}$$

The solutions for the parameters α and β can be represented in the form of

$$\alpha = M_1 e^{1/2 i\lambda\tau} + M_2 e^{-1/2 i\lambda\tau}, \qquad \beta = N_1 e^{1/2 i\lambda\tau} + N_2 e^{-1/2 i\lambda\tau} \ . \tag{10}$$

The integration constants M and N are related by the following dependences:

$$N_1 = -ik_1 M_1, \qquad N_2 = -ik_2 M_2 \ . \tag{11}$$

Here, the coefficients k_1 and k_2 are determined by the formulas

$$k_{1,2} = \left(\frac{\cot\nu}{C} \pm R\right)\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)^{-1} \quad \left(R = \left(\frac{\cot^2\nu}{C^2} + \frac{\cos^2\mu}{B^2} + \frac{\sin^2\mu}{A^2}\right)^{1/2}\right). \tag{12}$$

We again will bear in mind the condition

$$\alpha\bar{\alpha} + \beta\bar{\beta} = 1 \tag{13}$$

from which, after substituting into it the solutions of eq.(10) with consideration of eq.(11), we obtain

$$M_1\bar{M}_1(1 + k_1\bar{k}_1) + M_2\bar{M}_2(1 + k_2\bar{k}_2) = 1 \ . \tag{14}$$

Thus, the Cayley-Klein parameters α and β, expressed as functions of the angle of rotation of the controlling flywheel τ, have the form

$$\alpha = M_1 e^{1/2 i\lambda\tau} + M_2 e^{-1/2 i\lambda\tau}, \qquad \beta = -ik_1 M_1 e^{1/2 i\lambda\tau} - ik_2 M_2 e^{-1/2 i\lambda\tau} \ . \tag{15}$$

Since the moduli of the constants M_1 and M_2 satisfy eq.(14), the obtained solutions actually contain only three independent real constants of integration corresponding to three degrees of freedom at three-dimensional rotation of the body. Let us multiply the first of equations (15) in turn by k_1 and k_2 and then subtract from it each time the second equation. This will yield

$$(k_1 - k_2) M_2 e^{-1/2 i\lambda\tau} = k_1\alpha - i\beta, \qquad (k_2 - k_1) M_1 e^{1/2 i\lambda\tau} = k_2\alpha - i\beta \ . \tag{16}$$

From this, we directly obtain two complex integrals

$$(k_1\alpha - i\beta) e^{1/2 i\lambda\tau} = \text{const}, \qquad (k_2\alpha - i\beta) e^{-1/2 i\lambda\tau} = \text{const} \ . \tag{17}$$

To solve a problem in Euler angles, we must express α and β according to eqs.(2). The integrals of eq.(17) then take the form

$$\begin{aligned} &[k_1 \cos 1/2\theta e^{1/2 i(\psi+\varphi)} + \sin 1/2\theta e^{1/2 i(\psi-\varphi)}] e^{1/2 i\lambda\tau} = \text{const} \\ &[k_2 \cos 1/2\theta e^{1/2 i(\psi+\varphi)} + \sin 1/2\theta e^{1/2 i(\psi-\varphi)}] e^{-1/2 i\lambda\tau} = \text{const} \ . \end{aligned} \tag{18}$$

On multiplying the last two integrals and replacing k_1 and k_2 by their expressions, we obtain an integral which does not contain the argument τ:

$$\begin{aligned} &[C^{-1}\cot\nu \sin\theta - \cos\theta (B^{-1}\cos\varphi\cos\mu + A^{-1}\sin\varphi\sin\mu) - \\ &\quad - i(B^{-1}\sin\varphi\cos\mu - A^{-1}\cos\varphi\sin\mu)] e^{i\psi} = \text{const} \ . \end{aligned} \tag{19}$$

This integral contains all three Euler angles θ, ψ, φ. However, we can ob-

tain an integral containing only two Euler angles θ and φ, expressing, for example, the modulus of any of the integrals of eq.(18) which also is an integral of the problem

$$\cos\theta + \tan\nu \sin\theta \,(CB^{-1}\cos\varphi\cos\mu + CA^{-1}\sin\varphi\sin\mu) = \text{const}\,. \tag{20}$$

To accomplish rotation of a body from a certain starting position θ_0, /173 ψ_0, φ_0 to a required final position θ_1, ψ_1, φ_1, a suitable selection of the angles of placement of the axis of the flywheel μ and ν and also of the angle of its rotation τ is necessary. In this case, we must bear in mind that both the values of the moments of inertia A, B, C and the directions of the principal axes in the body, generally speaking, depend on the angles of installation of the flywheel μ and ν. However, at comparatively small dimensions of the flywheel, this dependence can be disregarded in practice. Then, to determine the angles μ and ν we can directly use the obtained integrals (19) and (20). The final formulas for determining the angles of setting the axes of the flywheel can be written

$$\begin{aligned} B\tan\mu = A\,[&\sin\Delta\psi\,(\cos\theta_1\cos\varphi_1\sin\theta_0 - \cos\theta_0\cos\varphi_0\sin\theta_1) + \\ &+ \cos\Delta\psi\,(\sin\varphi_1\sin\theta_0 + \sin\varphi_0\sin\theta_1) - \sin\theta_1\sin\varphi_1 - \sin\theta_0\sin\varphi_0]\times \\ &\times[\sin\Delta\psi\,(-\cos\theta_1\sin\varphi_1\sin\theta_0 + \cos\theta_0\sin\varphi_0\sin\theta_1) + \\ &+ \cos\Delta\psi\,(\cos\varphi_1\sin\theta_0 + \cos\varphi_0\sin\theta_1) - \sin\theta_1\cos\varphi_1 - \sin\theta_0\cos\varphi_0]^{-1} \end{aligned} \tag{21}$$

$$\begin{aligned} C\tan\nu = [&\cos\theta_0 - \cos\theta_1]\,[B^{-1}\cos\mu\,(\sin\theta_1\cos\varphi_1 - \sin\theta_0\cos\varphi_0) + \\ &+ A^{-1}\sin\mu\,(\sin\theta_1\sin\varphi_1 - \sin\theta_0\sin\varphi_0)]^{-1} \end{aligned} \tag{22}$$

$$\Delta\psi = \psi_1 - \psi_0\,.$$

Knowing μ and ν, we can easily calculate λ and $k_{1,2}$ after which, using any of the integrals (18), we can find the angle τ. Thus, these formulas permit determining the setting angles of the flywheel and the angle of its rotation τ which are needed for accomplishing the required reorientation of the supporting body.

To define the general character of motion of the supporting body under the

effect of rotation of an internal flywheel, we will again resort to the integrals (17). Equation (12) will be written in the form of

$$k_{1,2} = \left(1 \pm \frac{\cot \nu}{CR}\right)\left(\frac{\cos^2 \mu}{B^2} + \frac{\sin^2 \mu}{A^2}\right)^{-1/2} R \exp\left[-i \arctan\left(\frac{B}{A} \tan\mu\right)\right] . \qquad (23)$$

Let us introduce the new angles μ^* and ν^*, determined by the following formulas

$$\tan\mu^* = \frac{B}{A}\tan\mu, \qquad \cos\nu^* = \frac{c}{CR}\nu, \qquad \sin\nu^* = \frac{1}{R}\left(\frac{\cos^2\mu}{B^2} + \frac{\sin^2\mu}{A^2}\right)^{1/2} . \qquad (24)$$

Then, for the coefficients k_1 and k_2 we obtain

$$k_1 = \cot {}^1/_2\nu^* e^{-i\mu^*}, \qquad k_2 = -\tan {}^1/_2\nu^* e^{-i\mu^*} \qquad (25)$$

Substituting these expressions into the integrals (17) and multiplying them by the necessary constant multipliers, we can write them in the form

$$\begin{aligned} (\alpha \cos {}^1/_2\nu^* e^{-1/2 i\mu^*} - i\beta \sin {}^1/_2\nu^* e^{1/2 i\mu^*})\, e^{1/2 i\lambda\tau} &= \text{const} \\ (-\alpha i \sin {}^1/_2\nu^* e^{-1/2 i\mu^*} + \beta \cos {}^1/_2\nu^* e^{1/2 i\mu^*})\, e^{-1/2 i\lambda\tau} &= \text{const} \end{aligned} . \qquad (26)$$

The expressions in parentheses are the Cayley-Klein parameters of the total rotation, from a system of fixed axes to a system of axes connected with the supporting body but not coinciding with its principal axes of inertia. This latter system of axes, fixed in the body, can be called the associate system since one of its axes forming the angles μ^* and ν^* with the principal axes of inertia of the body complement the axes of the flywheel in the body; tan μ^* and tan ν^* are determined by eqs.(24).

The angle μ^*, although it will not be a Euler angle (angle of precession), actually differs from it by a constant magnitude. Therefore, denoting the Euler angles of the associate system of axes with respect to its fixed system by θ^*, ψ^*, φ^* and the complementary Cayley-Klein parameters by α^* and β^*, we can write the integrals (26) in the form of

$$\alpha^* e^{\frac{1}{2}i\lambda\tau} = \text{const}, \qquad \beta^* e^{-\frac{1}{2}i\lambda\tau} = \text{const} \tag{27}$$

or, in angles θ^*, ψ^*, φ^*,

$$\cos \tfrac{1}{2}\theta^* e^{\frac{1}{2}i(\psi^*+\varphi^*+\lambda\tau)} = \text{const}, \qquad \sin \tfrac{1}{2}\theta^* e^{\frac{1}{2}i(\psi^*-\varphi^*-\lambda\tau)} = \text{const} . \tag{28}$$

From this, it is immediately obvious that the angles θ^* and ψ^* are re- /174
tained as constants during motion, whereas the angle φ^* varies linearly as a function of τ, i.e.,

$$\theta^* = \theta_0^* = \text{const}, \qquad \psi^* = \psi_0^* = \text{const}, \qquad \varphi^* = \varphi_0^* - \lambda\tau . \tag{29}$$

Thus, during rotation of an internal flywheel about an arbitrarily fixed axis, the housing of the supporting body executes a pure rotation about another fixed axis whose direction in space is given by the angles θ^* and ψ^*, whereas in the body it is given by the angles μ^* and ν^*. We note that the previously derived integrals (19) and (20) express none other than the constant of the cosines of the angles between the axis of rotation of the body and the fixed coordinate axes. It is evident that the axis of rotation of the body coincides with the axis of the flywheel only if either A = B = C or if the axis of the flywheel is directed along one of the principal axes of inertia of the body.

According to the Euler theorem (Bibl.2), any reorientation of a solid can be kinematically satisfied directly by its rotation about a certain fixed axis. The obtained result indicates that such a rotation can be dynamically accomplished by means of one flywheel with a fixed axis. In order to find the required angles of setting the axis of the flywheel in the body with respect to a given initial and final position of the supporting body, we must determine, on the basis of kinematic constructions (Bibl.2), the axis of rotation of the supporting body, i.e., we must find the angles μ^* and ν^* and then, using eqs.(24), calculate the unknown angles μ and ν.

Thus, the rotation of the supporting body about an arbitrary axis under the

effect of rotation of an internal flywheel can be calculated from relationships analogous to the calculation of the rotation of the body about one of its principal axes of inertia, i.e., we can use equations of two-dimensional rotation in calculating three-dimensional rotation.

The last of the integrals in eqs.(29) will then have the form

$$J\tau + \varphi^* (C^2 \cos^2 \nu^* + B^2 \sin^2 \nu^* \cos^2 \mu^* + A^2 \sin^2 \nu^* \sin^2 \mu^*)^{1/2} = \text{const} , \qquad (30)$$

where the square root plays the role of a certain reduced moment of inertia of the supporting body relative to the axis of its rotation. It is obvious that the conclusion is valid only when the initial kinetic momentum of the supporting body-flywheel system is equal to zero.

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